Coherent States in Action[*]

John R. Klauder

Departments of Physics and Mathematics

University of Florida

Gainesville, Fl 32611

Abstract

Quantum mechanical phase space path integrals are re-examined with regard to the physical interpretation of the phase space variables involved. It is demonstrated that the traditional phase space path integral implies a meaning for the variables involved that is manifestly inconsistent. On the other hand, a phase space path integral based on coherent states entails variables that exhibit a self-consistent physical meaning.

1 Conventional phase space path integrals

There is considerable appeal in the formal phase space path integral

$$\langle q'' | \, e^{-i\mathcal{H}T} \, | q' \rangle = \mathcal{N} \int e^{i \int [p\dot{q} - h(p,q)] \, dt} \, \mathcal{D}p \, \mathcal{D}q \tag{1}$$

which yields the propagator in the q-representation [1]. In this relation the integration is over all q-paths $q(t)$, $t' \leq t \leq t'' \equiv t' + T$, $T > 0$, subject to the boundary conditions that $q(t'') = q''$ and $q(t') = q'$, as well as all p-paths $p(t)$ for $t' \leq t \leq t''$. It follows from this formula that the *meaning* of $q(t)$ is the same as the meaning of $q(t'')$, namely, as the sharp eigenvalue of the position operator Q, where $Q|q\rangle = q|q\rangle$.

An analogous path integral leads to the propagator in the p-representation and is given by

$$\langle p''| e^{-i\mathcal{H}T} |p'\rangle = \mathcal{N} \int e^{i\int[-q\dot{p} - h(p,q)]\,dt} \, \mathcal{D}p\,\mathcal{D}q \;. \tag{2}$$

In this expression integration runs over the p-paths $p(t)$, $t' \leq t \leq t''$, subject to the requirement that $p(t'') = p''$ and $p(t') = p'$, while in the present case, integration over all q-paths $q(t)$, $t' \leq t \leq t''$, is assumed. It follows that the *meaning* of $p(t)$ is the same as the meaning of $p(t'')$, namely as the sharp eigenvalue of the momentum operator P, where $P|p\rangle = p|p\rangle$.

These two path integrals are of course connected with each other. In particular, it follows that

$$
\begin{aligned}
\langle q''| e^{-i\mathcal{H}T} |q'\rangle &= (2\pi)^{-1} \int e^{i(q''p'' - q'p')} \langle p''| e^{-i\mathcal{H}T} |p'\rangle \, dp''\, dp' \\
&= \mathcal{N} \int e^{i\int[\dot{\overline{pq}} - q\dot{p} - h(p,q)]\,dt} \, \mathcal{D}p \,, \mathcal{D}q \\
&= \mathcal{N} \int e^{i\int[p\dot{q} - h(p,q)]\,dt} \, \mathcal{D}p\,\mathcal{D}q
\end{aligned}
\tag{3}
$$

just as before.

Is the so obtained physical meaning for $p(t)$ and $q(t)$ satisfactory? If we were dealing with the strictly classical theory, for which $\hbar = 0$, there is absolutely no contradiction in specifying $p(t)$ and $q(t)$ simultaneously for all t, $t' \leq t \leq t''$. On the other hand, we are dealing with the quantum theory and decidedly not the classical theory. Planck's constant $\hbar = 1$ (in the chosen units) and does not vanish. T hus we are led to the conclusion that the given formal path integrals are expressed in terms of phase space paths for which, *within the quantum theory*, one may simultaneously specify both position $q(t)$ and momentum $p(t)$, $t' < t < t''$ sharply. This assertion evidently contradicts the Heisenberg uncertainty principle, and consequently it is unacceptable. Something is definitely wrong!

Another indication that something is wrong follows on consideration of the expression

$$\mathcal{N} \int e^{i\int[\frac{1}{2}(p\dot{q} - q\dot{p}) - h(p,q)]\,dt} \, \mathcal{D}p\,\mathcal{D}q \;, \tag{4}$$

which also involves an acceptable version of the classical action, but which cannot be interpreted along the lines given above. Interpretation fails because

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it is unclear what variable(s) are to be held fixed at the initial and final times. For instance, should this expression be interpreted as

$$C \int e^{i(p''q''-p'q')/2} \langle p''| e^{-i\mathcal{H}T} |p'\rangle \, dp'' \, dp' \ , \tag{5}$$

where C is an appropriate constant, or as

$$C \int e^{-i(p''q''-p'q')/2} \langle q''| e^{-i\mathcal{H}T} |q'\rangle \, dq'' \, dq' \tag{6}$$

either of which would seem to be equally possible interpretations but which evidently lead to unequal results.

1.1 Why do interpretational problems exist?

The reason these expressions lead to inconsistencies of interpretation is really very simple—although it is a reason that physicists are often reluctant to entertain. The argument presented above fails because the *indicated representations* for $\langle q''|e^{-i\mathcal{H}T}|q'\rangle$ and $\langle p''|e^{-i\mathcal{H}T}|p'\rangle$ simply *do not exist* as given. Physicists tend to believe that if they can write down a set of relations possessing a *formal* self consistency, then the underlying existence of the relations is not in doubt.[1] Of course, the dilemma surrounding these relations can be lifted by giving alternative representations that, in fact, do exist. One such representation is based on a lattice limit, namely, by giving meaning to the undefined formal path integral as the limit of a sequence of well defined finite dimensional integrals. As one such prescription we offer [2, 3]

$$\langle q''| e^{-i\mathcal{H}T} |q'\rangle$$
$$= \lim_{N\to\infty} \frac{1}{(2\pi)^{N+1}} \int \exp\{i\Sigma_{l=0}^{N}[p_{l+1/2}(q_{l+1}-q_l) - \epsilon h(p_{l+1/2},(q_{l+1}+q_l)/2)]\}$$
$$\times \Pi_{l=0}^{N} \, dp_{l+1/2} \, \Pi_{l=1}^{N} \, dq_l \ . \tag{7}$$

Here the limit $N \to \infty$ also implies that $\epsilon \equiv T/(N+1) \to 0$, and $q_{N+1} \equiv q''$ and $q_0 \equiv q'$; all p values are integrated out. This prescription, which

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[1]A simple but informative example of this issue is the following. Let $\{1,2,3,...\}$ denote the set of positive integers. Let X denote the largest such integer, and let us assume that $X > 1$. Since $X^2 > X$, we observe there is an integer larger than X, therefore we conclude that our assumption that $X > 1$ was in error, hence $X = 1$.

applies for a wide class of classical Hamiltonian functions $h(p, q)$, generates the propagator in the Schrödinger q-representation, and two such propagators properly fold to a third propagator when integrated over the intermediate q with a measure dq.

However, this is not the only prescription that can be offered for the same formal phase space path integral.

2   Coherent state formulation

Another rule of definition that can also be accepted for the formal phase space path integral (1) is given by [4]

$$
\begin{aligned}
\langle p'', & q'' | \, e^{-i\mathcal{H}T} \, | p', q' \rangle \\
&\equiv \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int \exp\big(\Sigma_{l=0}^N \{ i\tfrac{1}{2}(p_{l+1} + p_l)(q_{l+1} - q_l) \\
&\qquad\qquad - \tfrac{1}{4}[(p_{l+1} - p_l)^2 + (q_{l+1} - q_l)^2] \\
&\qquad\qquad - i\epsilon h(\tfrac{1}{2}(p_{l+1} + p_l + iq_{l+1} - iq_l), \tfrac{1}{2}(q_{l+1} + q_l - -ip_{l+1} + ip_l)) \} \big) \\
&\qquad\qquad \times \Pi_{l=1}^N \, dp_l \, dq_l \ .
\end{aligned}
\tag{8}
$$

This expression differs from the former one in that both p *and* q are held fixed at the initial *and* final end points. In particular, now $(p_{N+1}, q_{N+1}) \equiv (p'', q'')$ and $(p_0, q_0) \equiv (p', q')$. Two such propagators properly fold together to a third propagator with an integration over the intermediate variables p and q with the measure $dp \, dq/2\pi$. Like the previous case, the present expression holds for a wide class of classical Hamiltonian functions. However, this latter sequence is *fundamentally* different than the previous one, and that difference not only involves a different sort of representation but even goes so far as to entail a profound change of the *meaning* of the symbols p and q from their meaning as found in the preceding section.

The states $|p, q\rangle$ implicitly introduced above are *canonical coherent states* defined by the following expression

$$
|p, q\rangle \equiv e^{-iqP} e^{ipQ} \, |0\rangle \ ,
\tag{9}
$$

where, as usual, $[Q, P] = i\mathbb{1}$ and $|0\rangle$ denotes the ground state of a harmonic oscillator, i.e., a normalized solution of the equation $(Q + iP)|0\rangle = 0$ [5].

Observe in this case that neither p nor q are *eigenvalues* of any operator. Instead, it follows that

$$\langle p, q|P|p, q\rangle = p , \qquad\qquad \langle p, q|Q|p, q\rangle = q , \qquad (10)$$

namely, that the labels p and q have the meaning of *expectation values* rather than eigenvalues. Thus there is absolutely no contradiction with the Heisenberg uncertainty principle in specifying both p and q *simultaneously*. The overlap of two coherent states, given by

$$\langle p', q'|p, q\rangle = \exp\{i\tfrac{1}{2}(p' + p)(q' - q) - \tfrac{1}{4}[(p' - p)^2 + (q' - q)^2]\} , \qquad (11)$$

serves as a reproducing kernel for the functional Hilbert space representation in the present case. The folding of two such overlap functions leads to

$$\int \langle p'', q''|p, q\rangle\langle p, q|p', q'\rangle \, dp \, dq/2\pi = \langle p'', q''|p', q'\rangle , \qquad (12)$$

an expression which shows that the coherent state overlap function serves as the "δ-function" in the present representation although, of course, in the present case it is a bounded, continuous function. In short, we learn that the choice of sequential definition adopted to give meaning to the formal phase space path integral can lead to a dramatic change of representation and even of the meaning of the variables involved.

We conclude these remarks with the observation that if we formally interchange the limit and integrations in (8) and write for the integrand the form it assumes for continuous and differential paths, the result has the formal expression (1), namely

$$\mathcal{N} \int e^{i \int [p\dot{q} - h(p, q)] \, dt} \, \mathcal{D}p \, \mathcal{D}q , \qquad (13)$$

which is just the expression we started with! It is in this sense that we assert that the present sequential definition is just as valid as the one customarily chosen. Moreover, with the present understanding of the sequential definition, there is absolutely no conflict between the meaning of the variables p and q and the Heisenberg uncertainty principle; in the present case, p and q denote expectation values in the coherent states involved, and these may both be specified as general functions of time $p(t)$ and $q(t)$, $t' \leq t \leq t''$.

It is clear to this author—but apparently unclear to many others—that the interpretation of the formal path integral (13) in terms of paths $p(t)$ and $q(t)$ for which the meaning of the variables is that of *expectation values* is far more acceptable than the one in which the meaning is that of both sharp *position* and sharp *momentum* (eigen)values. Even if one carries to the continuum the insight gained on the lattice for the usual formulation, namely, that p and q are diagonalized *alternately* on successive time slices, the result is that the continuum interpretation is strictly not one for which p and q are *simultaneously* sharp but one where p and q are alternately sharp at every "other" instant of time—and of course when $p(q)$ is sharp then $q(p)$ is completely unknown! This is the true physical meaning of the variables entering the putative formal phase space path integral with the usual interpretation. How bizarre that interpretation is when it is fully appreciated for what it is!

Contrast the interpretation just outlined with the one appropriate to the alternative scenario in terms of canonical coherent states. In the case of a lattice formulation of the phase space path integral in terms of coherent states, p and q are specified at each time slice simultaneously and interpreted as expectation values. This interpretation persists in the continuum limit, and there is no logical conflict of that interpretation in such a limit. Moreover, there is a symmetry in the interpretation and usage of p and q inherent in the coherent state formulation that is simply unavailable in the more traditional formulation.

One is almost tempted to assert that the usual interpretation in terms of sharp eigenvalues is "wrong", because it cannot be consistently maintained, while the interpretation in terms of expectation values is "right", because it can be consistently maintained. On the other hand, the community at large may not be ready to swallow such a heretical statement, so perhaps it would be best if it was stricken from the record! However, before completely striking it from the record, it may not be inappropriate to offer additional evidence as food for thought.

3   Wiener measure regularization

We have accepted the fact that (13) is without mathematical meaning as it stands. Some sort of regularization and removal of that regularization is needed to give it meaning. There are many ways to do so, two of which have

been illustrated above. In this section we discuss quite a different form of regularization.

Consider the expression [6]

$$\lim_{\nu \to \infty} \mathcal{N} \int \exp\{i \int [p\dot{q} - h(p,q)]\, dt\} \, \exp[-(1/2\nu) \int (\dot{p}^2 + \dot{q}^2)\, dt]\, \mathcal{D}p\, \mathcal{D}q \ . \quad (14)$$

This expression differs from the usual one (13) by the presence of a damping factor—a convergence factor—involving the time derivative of both p and q. The result of interest is given in the limit that the parameter $\nu \to \infty$. Note that when $\nu = \infty$, formally speaking, the usual formal path integral (13) is recovered. Although (14) is written in the same formal language as (13), the latter expression is in fact profoundly different. In fact, (14) is intended to be a *regularized* form of (13). Admittedly, it doesn't appear any better defined than the usual expression in its present form; however, (14) can be given an alternative but equivalent formulation when we group certain terms together. In particular, with a suitable regrouping of terms (14) becomes

$$\lim_{\nu \to \infty} (2\pi)\, e^{\nu T/2} \int e^{i \int [p\, dq - h(p,q)\, dt]}\, d\mu^\nu_W(p,q) \ . \quad (15)$$

In this expression μ^ν_W denotes (pinned) Wiener measure on the two dimensional plane expressed in Cartesian coordinates (p,q). In addition, $p(t)$ and $q(t)$, $t' \le t \le t''$, denote Brownian motion paths with ν as the diffusion constant, and $\int p\, dq$ denotes a stochastic integral needed since although $p(t)$ and $q(t)$ are continuous functions for all ν they are nowhere differentiable. For convenience we adopt the Stratonovich (midpoint) definition of the stochastic integral (which is equivalent to the Itô definition in the present case because $dp(t)dq(t) = 0$ is a valid Itô rule in these coordinates). With those remarks the integral in (15) is a well defined expression for each ν and one may ask the question whether the indicated limit converges and if so whether that limit has anything to do with a solution to the Schrödinger equation. For a dense set of Hamiltonians the answer to both of these questions is *yes!*

However, before we relate this expression to the earlier discussion let us take up the possible meaning of the variables p and q as they appear in (15). Observe, as noted, that the expression is well defined as it stands—indeed, it involves a *continuous time regularization*. Thus if this expression is going to have something to do with quantum mechanics it must be consistent to simultaneously specify both $p(t)$ and $q(t)$ for all t in the appropriate interval. This means that p and q *cannot* have the meaning of sharp momentum

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and sharp position, respectively. On the other hand, it would be possible for those variables to have the meaning of expectation values which can be simultaneously given. It should thus come as not too great a surprise that the continuous time regularization of a phase space path integral with the help of a Wiener measure on the plane, in the limit as the diffusion constant diverges, *automatically generates a coherent state representation!*

In particular, with the Brownian paths pinned so that $p(t'') = p''$, $q(t'') = q''$ and $p(t') = p'$, $q(t') = q'$, the resultant limit is equivalent to

$$\langle p'', q'' | e^{-i\mathcal{H}T} | p', q' \rangle$$
$$= \lim_{\nu \to \infty} (2\pi) \, e^{\nu T/2} \int e^{i \int [p \, dq - h(p,q) \, dt]} \, d\mu_W(p,q) \,, \qquad (16)$$

where, as implied by (16) itself, and consistent with the earlier notation,

$$|p, q\rangle \equiv e^{-iqP} e^{ipQ} |0\rangle \,, \qquad (Q + iP)|0\rangle = 0 \,, \qquad (17)$$
$$\mathcal{H} \equiv \int h(p, q)|p, q\rangle\langle p, q| \, dp \, dq/2\pi \,. \qquad (18)$$

In other words, the result of the Wiener measure regularized phase space path integral, in the limit that the diffusion constant diverges, yields a propagator in the coherent state representation as we had discussed earlier. Here is an additional argument for favoring the interpretation of the formal phase space path integral as really standing for the one expressed in terms of coherent states rather than one that is internally inconsistent, namely, one interpreted in terms of sharp eigenvalues for the position and momentum.

If one accepts the idea that the formal expression (13) may be best interpreted in terms of coherent states rather than sharp Schrödinger eigenstates, one may be worried that many previous calculations are incorrect. There is no need to worry. All previous calculations which are implicitly consistent with a lattice limit such as in (8) are perfectly correct. Our discussion is not addressed to revising the evaluation of properly interpreted path integrals but rather to stressing the consistency—or possible inconsistency—of interpreting the continuum version of the phase space path integral. With the coherent state interpretation one is completely justified in regarding the paths as functions defined for a continuous time parameter, and indeed within the sequence where $\nu < \infty$, as continuous functions of time. This is a *conceptual* difference with respect to how the interpretation in the usual formulation can be taken. If there is ever any hope to define path integrals rigorously as

path integrals over a set of paths (functions of time), then it is *essential* to give up the notion that the paths involved are sharp value paths and replace that with another interpretation of which the expectation value paths is a completely satisfactory example. In point of fact, the present author feels that the rigorous definition (16) in terms of a limit of a sequence of *completely unambiguous* path integrals is as close as one is likely to come to a rigorous definition of a continuous time path integral in terms of genuine (i.e., countably additive) measures. One can hardly ask for an expression without some sort of regularization. For example, even the one dimensional integral

$$\int_{-\infty}^{\infty} e^{iy^2} \, dy \tag{19}$$

is effectively undefined since it is only conditionally convergent. It, too, needs a rule to overcome this ambiguity, and one rule is to define it as

$$\lim_{\nu \to \infty} \int_{-\infty}^{\infty} e^{iy^2 - y^2/\nu} \, dy \ . \tag{20}$$

The indicated sequence exists and the limit converges, but it has required the use of a convergence factor; one could hardly expect a real time path integral to require anything less!

3.1 Generalization to non-flat phase spaces

The point we are making here naturally leads to another line of thought on which we shall comment but not develop since it has been adequately treated elsewhere. If we are dealing with a conditionally convergent integral, then it is possible to obtain fundamentally different answers by choosing a qualitatively different form of regularization. In particular, from the point of view of regularization, why was it necessary for us to choose a Brownian motion regularization on a phase space that constitutes a *flat* two-dimensional space; why not consider a Brownian motion regularization on a phase space that is a curved two-dimensional manifold, say, a sphere or a pseudo-sphere, for example, or even a space of non-constant curvature. Brownian motion regularization on such non-flat spaces has indeed been investigated, and the result is most interesting. For a sphere (of the right radius) the result of the limit of the regulated phase space path integrals over continuous paths leads to a quantization in which the kinematical variables are spin variables, i.e.,

operators that obey the commutation relations of the Lie algebra of the group SU(2) [6]. If a pseudo-sphere is used instead, the result for the kinematical variables is that for the Lie algebra of the group SU(1,1) (or the "$ax + b$" group) [7, 8]. Both of these cases lead to group related coherent states and a representation of the propagator in terms of those states. On the other hand, for Brownian motion on a space without any special symmetry, the result again leads to coherent states [9], but these are coherent states of a more general kind than traditionally considered since they are *not* associated with any group!

The moral of this extended story is that phase space path integrals of an exceedingly general kind appropriate to very general kinematical variables can be rigorously developed with the aid of a Weiner measure regularization each of which involves coherent states wherein the variables are *never* eigenvalues of some self-adjoint operator but more typically are associated with expectation values of suitable operators for which there is no conceptual difficulty in their simultaneous specification. This very desirable state of affairs has arisen by combining the *symplectic* geometry of the classical theory with a *Riemannian* geometry needed to carry the *Brownian motion* that forms the regularization.

If we may be allowed a single phrase of summary, then it is no exaggeration to claim [10] that, when properly interpreted,

$$\text{QUANTIZATION} = \text{GEOMETRY} + \text{PROBABILITY}$$

4   Acknowledgments

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